SET THEORY HOMEWORK 5

Due Wednesday, November 20.

Problem 1. *Suppose that $\mathbb{P}, \mathbb{Q}$ are two posets in the ground model V and $\pi : \mathbb{P} \to \mathbb{Q}$ is a projection, i.e.*

1. *π is order preserving: $p' \leq p \to \pi(p') \leq \pi(p)$,*
2. *for all $p \in \mathbb{P}$ and $q \leq \pi(p)$, there is $p' \leq p$, such that $\pi(p') \leq q$.*

Suppose that G is a $\mathbb{P}$-generic filer over V. Show that $H := \{q \mid \exists p \in G, \pi(p) \leq q\}$ is a $\mathbb{Q}$-generic filter over V.

Problem 2. *Suppose $\mathbb{P}$ and $\mathbb{Q}$ are two posets and $i : \mathbb{P} \to \mathbb{P}$ is such that:*

1. *$i(1_{\mathbb{P}}) = 1_{\mathbb{Q}}$;*
2. *If $p' \leq p$, then $i(p') \leq i(p)$;*
3. *For all $p_1, p_2 \in \mathbb{P}$, $p_1 \perp p_2$ iff $i(p_1) \perp i(p_2)$;*
4. *If A is a maximal antichain of $\mathbb{P}$, then i"$A := \{i(p) \mid p \in A\}$ is a maximal antichain in $\mathbb{Q}$.*

Suppose also that H is $\mathbb{Q}$-generic. Show that $G := \{p \in \mathbb{P} \mid i(p) \in H\}$ is $\mathbb{P}$-generic and that $V[G] \subset V[H]$, where V is the ground model.

Remark: an embedding as above is called **a complete embedding**

Problem 3. *Suppose that $\mathbb{P}$ is a poset, $A \subset \mathbb{P}$ is a maximal antichain, $\phi(x)$ is a formula, and $\langle \tau_p \mid p \in A\rangle$ are $\mathbb{P}$ names such that for all $p \in A$, $p \Vdash \phi(\tau_p)$. Then there is a $\mathbb{P}$ name τ, such that $1_{\mathbb{P}} \Vdash \phi(\tau)$.*

Problem 4. *Suppose that κ is measurable with a normal measure U and $\mathbb{P}$ is the Prikry forcing at κ with respect to U.*

1. *Suppose that G is $\mathbb{P}$-generic and let $\langle \alpha_n \mid \alpha_n\rangle = \bigcup_{\langle s,A\rangle \in G} s$. Show that for all $A \in U$, for all large n, $\alpha_n \in A$.*
2. *Suppose that $\vec{\alpha}\langle \alpha_n \mid \alpha_n\rangle$ is an increasing cofinal sequence through κ (not in V), such that for all $A \in U$, $A \in V$, for all large n, $\alpha_n \in A$. Show that $G := \{\langle s, A\rangle \mid s = \{\alpha_i \mid i < |s|\}, \{\alpha_i \mid |s| \leq i < \omega\} \subset A\}$ is generic for $\mathbb{P}$.*

Problem 5. *Suppose κ is a regular cardinal, $\mathbb{P}$ is κ-c.c., and*

$$1_{\mathbb{P}} \Vdash \text{“}\dot{B} \text{ is a bounded subset of } \kappa.\text{”}$$

Show that some some $\gamma < \kappa$, $1_{\mathbb{P}} \Vdash$ “$\dot{B} \subset \gamma$”.